Interaction of a shock wave with a mixing region

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The interaction of a simple wave, in steady supersonic flow, with a two-dimensional mixing region is treated by applying Fourier analysis to the linearized equations of motion. From asymptotic forms for the Fourier transforms of physical quantities, for large wave-number, the dominant features of the resulting flow pattern are predicted; in particular it is found that a shock wave, incident on the mixing region, is reflected as a logarithmically infinite ridge of pressure. For two particular Mach-number distributions in the undisturbed flow, numerical solutions are obtained, showing greater detail than the results predicted by the asymptotic approach. A method is given whereby the linear theory may be improved to take into account some non-linear effects; and the reflected wave, for an incident shock wave, is then seen to consist of a shock wave, gradually diminishing in strength, followed by the main expansion wave.

1. Introduction

The problem of the interaction of a shock wave with a boundary layer has, over the past few years, received considerable attention from both experimental and theoretical workers. By contrast, the problem of the interaction of a shock wave with a mixing region has received scant attention. Perhaps the most common interaction of the latter type occurs when a supersonic jet emerges from a nozzle in which the pressure is less than atmospheric pressure, although an equally common but less obvious case is when a strong shock wave (i.e. strong enough to cause separation) interacts with a boundary layer. The separation point is then situated well upstream from the point of incidence of the initial shock which, consequently, is itself directly incident upon a layer of fluid in which the fluid velocity falls to zero, in the 'dead-air bubble', before the wall is reached, just as in a mixing region. The present problem can then perhaps also be considered as a special case of a shock-boundary-layer interaction. The main experimental results concerning interactions of shock waves with mixing regions are in the form of Schlieren photographs of supersonic jets, such as those taken at the Manchester University Fluid Motion Laboratory by Johannesen (1957). The difficulties in obtaining measurements by mechanical methods in these small regions are very great, owing to the relatively large disturbance created by any instrument which might be placed in the flow.

Important theoretical contributions to the problem of the interaction of a shock wave with a boundary layer have been made by Lighthill (1950, 1953). He considers an ideal fluid in which viscosity and heat conduction are neglected and replaces the boundary layer by a shear layer in which the Mach number falls from its main stream value M_1 to zero (Lighthill 1950) at the wall, or to M_2 (Lighthill 1953) when an inner viscous sublayer is considered. The equations of plane steady flow are used and products of small quantities arising from small disturbances to the basic parallel flow are neglected. In the present problem, concerning the interaction of a shock wave with a mixing region, the same procedure is adopted, the mixing region being represented by a semi-infinite layer of vorticity in which the Mach number varies continuously from its main stream value M_1 to zero asymptotically at large distances from the main stream. The flow is assumed to be parallel everywhere. Owing to the absence of any solid boundary, there is no need to consider any inner viscous sublayer in this problem.

Having substituted this shear layer for the mixing region, it would hardly seem fair to refer to the mixing region as such, as in fact no mixing is taking place; consequently the 'mixing region' will be referred to as the 'non-uniform' region. The equations of motion are formulated in terms of derivatives $\partial/\partial s$ and $\partial/\partial n$ along and at right angles to the streamlines, with the flow direction and a suitably defined pressure coefficient as dependent variables depending only on the Mach-number distribution, and not on the separate velocity and temperature distributions. If the disturbances to the uniform parallel flow are small and if, in the undisturbed state, the fluid is flowing in a direction parallel to the x-axis, then to a first approximation the derivatives $\partial/\partial s$ and $\partial/\partial n$ may be replaced by $\partial/\partial x$ and $\partial/\partial y$ and the Mach-number distribution by its undisturbed value.

The Fourier transform of the pressure coefficient is defined and an ordinary differential equation which it satisfies is derived. Asymptotic solutions of this equation, with a general undisturbed Mach-number distribution, are obtained for large wave-number in the subsonic region, the solutions being continued across the sonic line by a method due to Langer (1931). In this way asymptotic forms of the Fourier transform of the pressure coefficient, reflected wave and flow direction are obtained, which in turn enable us to pick out the singularities in these quantities. It is thus shown that an incident wave in the form of a shock wave is reflected as a pressure ridge, as in Lighthill's boundary-layer theory.

Using a specific Mach-number distribution in the non-uniform region, it has been found possible to solve the equation for the Fourier transform of the pressure coefficient exactly. Using this solution it is not difficult to show that the upstream influence (i.e. the distance upstream which the disturbance penetrates) is only of the order of one layer thickness. The reflected wave for this Mach-number distribution has been calculated using numerical methods, analytical methods proving too intractable. When the incident wave is in the form of a shock wave the reflected wave is seen to consist of a very narrow region of compression followed by a broader region of expansion; photographic evidence in support of this is available. From the reflected wave for an incident shock, it is shown to be possible to calculate the reflected wave for an incident Prandtl–Meyer expansion fan which is found to consist of a slight further expansion followed by a region of compression. An interaction of the latter type occurs when a supersonic jet, in which the pressure is greater than atmospheric pressure, exhausts into the atmosphere.

In the linear theory, first approximations to the physical quantities are obtained, but the characteristic network is left unchanged as two sets of parallel lines. This is the fundamental failure of linear theory, since the characteristics along which the physical quantities are predicted are incorrectly placed. Several methods (Whitham 1952; Lighthill 1957; and Kantrowitz 1958) are available for obtaining a correctly modified first approximation to the flow pattern, and the method used here, anticipated by both Lighthill and Kantrowitz, is outlined in an appendix. The flow is assumed isentropic, which means that third-order terms due to entropy changes at the shock are neglected, so that any shock waves occurring must not be too strong. The mass flow in a given direction is examined and in the isentropic-flow theory this becomes multi-valued under a compressive disturbance. It is shown that a shock wave, inserted so as to leave the total mass flow unaltered, renders the mass flow single-valued though discontinuous and satisfies all the equations governing the complete flow pattern, under the isentropic-flow limitation. If we were to make the further assumption that products of small quantities are to be neglected, we should recover the method outlined by Whitham (1952). The resulting flow patterns, displayed graphically, show very clearly how the reflected expansion wave interacts with the adjacent shock wave, gradually weakening it.

2. Equations of motion

The equations of motion governing a perfect gas, with constant adiabatic index γ and zero viscosity and thermal conductivity, which flows in a steady two-dimensional pattern, are

$$\left.\begin{aligned} \frac{\partial}{\partial s}(\rho u)+\rho u\frac{\partial \theta}{\partial n} &= 0,\\ \rho u\frac{\partial u}{\partial s} = -\frac{\partial p}{\partial s},\quad \rho u^2\frac{\partial \theta}{\partial s} &= -\frac{\partial p}{\partial n},\\ \frac{\partial p}{\partial s} &= a^2\frac{\partial \rho}{\partial s},\end{aligned}\right\} \tag{1}$$

expressing conservation of mass, momentum and entropy along a streamline, respectively. The derivatives $\partial/\partial s$ and $\partial/\partial n$ are derivatives along the streamlines and normal to them, respectively. The velocity is denoted by its magnitude u and direction θ which is the angle a streamline makes with some fixed direction, say the x-axis. The pressure p and density ρ are related to a the local velocity of sound by $a = (\gamma p/\rho)^{\frac{1}{2}}$. Elimination of $\partial u/\partial s$, $\partial\rho/\partial s$ between the first two and last of equations (1) gives

$$(1-M^2)\frac{\partial p}{\partial s} = \rho u^2\frac{\partial \theta}{\partial n}, \tag{2}$$

where $M = u/a$ is the local Mach number. If we now define a pressure coefficient $\phi = \log(p/p_1)^{1/\gamma}$, where p_1 is some constant pressure, then equation (2) and the third of equations (1) reduce to the simple form

$$\left.\begin{aligned}(1-M^2)\frac{\partial \phi}{\partial s} &= M^2\frac{\partial \theta}{\partial n},\\ \frac{\partial \phi}{\partial n} &= -M^2\frac{\partial \theta}{\partial s}.\end{aligned}\right\} \tag{3}$$

Suppose the flow pattern consists of a basic parallel flow in the x-direction upon which is superimposed a small disturbance, then the streamlines will only be displaced by a small amount from their undisturbed state, and so to a first approximation we may take $\partial/\partial s = \partial/\partial x$ and $\partial/\partial n = \partial/\partial y$, giving

$$\left.\begin{aligned} \{1-M^2(y)\}\frac{\partial\varpi}{\partial x} &= M^2(y)\frac{\partial\theta}{\partial y}, \\ \frac{\partial\varpi}{\partial y} &= -M^2(y)\frac{\partial\theta}{\partial x}. \end{aligned}\right\} \tag{4}$$

If p_1 is taken to be the pressure in the basic flow then the pressure coefficient $\varpi = (p-p_1)/\gamma p_1$. In the basic flow we assume that the Mach number distribution $M(y)$ is a function of y only, continuous for all y and decreasing in $0 \geqslant y > -\infty$ with $M(y) \to 0$ as $y \to -\infty$ and $M(y) = M_1 > 1$ for $y \geqslant 0$. We see then that the distribution of pressure coefficient ϖ and the streamline pattern (deducible from θ) depend only on the distribution of Mach number $M(y)$ and not on the separate velocity and temperature distributions. The equation for ϖ obtained by eliminating θ from equations (4) is

$$\frac{\partial^2\varpi}{\partial y^2} - 2\frac{M'(y)}{M(y)}\frac{\partial\varpi}{\partial y} + \{1-M^2(y)\}\frac{\partial^2\varpi}{\partial x^2} = 0. \tag{5}$$

If the cause of the disturbance is taken as a simple plane wave incident on the non-uniform region from outside, then, by the theory of small disturbances to a uniform supersonic stream, ϖ must, in $y > 0$, take the form

$$\varpi = f(x+\beta y) + g(x-\beta y), \tag{6}$$

where f represents the incident wave, g the reflected wave, and $\beta = (M_1^2-1)^{\frac{1}{2}}$. The function g is what we particularly wish to determine in the present investigation.

The boundary condition at $y = 0$ is that the pressure field be continuous with the field given by (6); this may be written

$$\left(\beta\frac{\partial\varpi}{\partial x} + \frac{\partial\varpi}{\partial y}\right)_{y=0} = 2\beta f'(x), \tag{7}$$

providing that the derivatives are continuous across $y = 0$. Lighthill (1950) has shown that they will be so as long as $M(y)$ is continuous. To complete the formulation of the problem we need another boundary condition, and it would appear reasonable to take this as

$$(\varpi)_{y\to-\infty} \to 0. \tag{8}$$

Equation (5) must then be solved under conditions (7) and (8). When the pressure field has been determined the reflected wave can be derived from the equation

$$\left(\beta\frac{\partial\varpi}{\partial x} - \frac{\partial\varpi}{\partial y}\right)_{y=0} = 2\beta g'(x), \tag{9}$$

and the flow direction θ from the second of equations (4).

3. Fourier analysis

If $f(x)$ can be expressed as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk, \tag{10}$$

then we shall seek solutions for ϖ, g and θ in the form

$$\varpi(x,y) = \int_{-\infty}^{\infty} e^{ikx} \Pi(k,y)\, dk, \tag{11}$$

$$g(x) = \int_{-\infty}^{\infty} e^{ikx} G(k)\, dk, \tag{12}$$

$$\theta(x,y) = \int_{-\infty}^{\infty} e^{ikx} H(k,y)\, dk. \tag{13}$$

If a substitution for ϖ, from (11), is made in equation (5), the following ordinary differential equation for Π is obtained:

$$\frac{d^2\Pi}{dy^2} - 2\frac{M'(y)}{M(y)}\frac{d\Pi}{dy} + k^2\{M^2(y)-1\}\Pi = 0, \tag{14}$$

the boundary conditions (7) and (8) becoming

$$\Pi_y(k,0) + ik\beta\Pi(k,0) = 2\beta ikF(k), \tag{15}$$

$$\Pi(k,y) \to 0 \quad \text{as} \quad y \to -\infty. \tag{16}$$

For convenience, when model profiles are chosen, solutions of equation (14) are better expressed in terms of the basic solution $\Pi_0(k,y)$, where

$$\Pi_0(k,0) = 1, \quad \Pi_0(k,y) \to 0 \quad \text{as} \quad y \to -\infty. \tag{17}$$

Then, to satisfy the boundary conditions (15) and (16), we must have

$$\Pi(k,y) = \frac{2\beta ikF(k)\Pi_0(k,y)}{\Pi_{0_y}(k,0) + ik\beta\Pi_0(k,0)}, \tag{18}$$

and from (9), (11), (12) and (18) the Fourier transform of the reflected wave is given as

$$\begin{aligned} G(k) &= [ik\beta\Pi(k,0) - \Pi_y(k,0)]/2\beta ik \\ &= -F(k)\frac{\Pi_{0_y}(k,0) - ik\beta\Pi_0(k,0)}{\Pi_{0_y}(k,0) + ik\beta\Pi_0(k,0)}. \end{aligned} \tag{19}$$

Also, we have the Fourier transform of the flow deflexion, from the second of equations (4) and equations (11), (13) and (18), given as

$$\begin{aligned} H(k,y) &= \frac{i}{kM^2(y)}\Pi_y(k,y) \\ &= -\frac{2\beta F(k)\Pi_{0_y}(k,y)}{M^2(y)\{\Pi_{0_y}(k,0) + ik\beta\Pi_0(k,0)\}}. \end{aligned} \tag{20}$$

Except for very special Mach-number distributions $M(y)$, equation (14) is very difficult to solve analytically. For the moment we shall confine ourselves to approximate solutions of equation (14) for large values of the parameter k, for, by now, it is well known how the singularities of a Fourier integral can be deduced from the asymptotic behaviour of the integrand (see e.g. Lighthill 1958). The main features of the flow can be predicted in this way. Solutions for a particular form of $M(y)$ will be discussed in § 5.

4. Asymptotic solutions

In order to estimate the asymptotic behaviour of $\Pi(k, y)$, we must refer to some of the results of a theory first fully developed by Langer (1931). The results which we shall need are set out briefly in appendix A; a more detailed summary is given by Lighthill (1950).

Considering now equation (14), we know from the 'W.B.K.' theory that asymptotic solutions as $|k| \to \infty$ must be a combination of the two forms

$$M(y)\{1-M^2(y)\}^{-\frac{1}{4}}\, e^{\pm iks}, \tag{21}$$

where

$$s = \int_{y_1}^{y} \{M^2(y)-1\}^{\frac{1}{2}}\, dy, \tag{22}$$

and $y = y_1$ is the sonic line in the undisturbed flow such that $M(y_1) = 1$. In the subsonic region, i.e. in the interval $-\infty < y < y_1$, we have $M(y) < 1$; hence $s = -i|s|$ from (22) and, from (21), any asymptotic solution as $|k| \to \infty$ must be a combination of

$$M(y)\{1-M^2(y)\}^{-\frac{1}{4}}\, e^{\pm k|s|}, \tag{23}$$

where $|s| \to \infty$ as $y \to -\infty$. Clearly then, to satisfy the second of conditions (17) we must take for the asymptotic form of $\Pi_0(k, y)$, as $|k| \to \infty$, in the subsonic region

$$\begin{aligned} \Pi_0(k,y) &\sim A_1 M(y)\{1-M^2(y)\}^{-\frac{1}{4}} \exp[-k|s|\operatorname{sgn} k] \\ &= A_1 M(y)\{1-M^2(y)\}^{-\frac{1}{4}}[H(k)\, e^{-k|s|} + H(-k)\, e^{k|s|}], \end{aligned} \tag{24}$$

where $\operatorname{sgn} k = k/|k|$ and $H(k) = 0$ for $k < 0$, $H(k) = 1$ for $k > 0$, and A_1 is, as yet, an arbitrary constant.

Writing $\Pi_0(k, y)$ as a combination $f_1(k)\, u_1(y) + f_2(k)\, u_2(y)$, where u_1 and u_2 are given in equation (62) by Langer's theory, enables us to determine $f_1(k)$ and $f_2(k)$ and hence continue the solution into the supersonic region $y > y_1$. Thus, since $s = -i|s|$,

$$f_1(k)\left(\frac{e^{k|s|}}{k^{\frac{1}{6}}} + \frac{e^{-k|s|}}{(-k)^{\frac{1}{6}}}\right) - f_2(k)\left(\frac{e^{k|s|}}{k^{\frac{5}{6}}} + \frac{e^{-k|s|}}{(-k)^{\frac{5}{6}}}\right) = A_1[H(k)\, e^{-k|s|} + H(-k)\, e^{k|s|}],$$

giving

$$\left.\begin{aligned} f_1(k) &= \frac{A_1}{\sqrt{3}}[H(k)\, k^{\frac{1}{6}} + H(-k)\,(-k)^{\frac{1}{6}}], \\ f_2(k) &= \frac{A_1}{\sqrt{3}}[H(k)\, k^{\frac{5}{6}} + H(-k)\,(-k)^{\frac{5}{6}}]. \end{aligned}\right\} \tag{25}$$

With $f_1(k), f_2(k)$ given by (25), we have

$$f_1(k)\left(\frac{e^{iks}}{(ik)^{\frac{1}{6}}} + \frac{e^{-iks}}{(-ik)^{\frac{1}{6}}}\right) + f_2(k)\left(\frac{e^{iks}}{(ik)^{\frac{5}{6}}} + \frac{e^{-iks}}{(-ik)^{\frac{5}{6}}}\right) = 2A_1 \cos(ks - \tfrac{1}{4}\pi \operatorname{sgn} k), \tag{26}$$

giving, as the solution in the supersonic region, as $|k| \to \infty$,

$$\Pi_0(k, y) \sim \frac{M(y)\{M^2(y)-1\}^{-\frac{1}{4}} \cos(ks - \frac{1}{4}\pi \operatorname{sgn} k)}{M_1(M_1^2-1)^{-\frac{1}{4}} \cos(k\sigma - \frac{1}{4}\pi \operatorname{sgn} k)}, \tag{27}$$

where $\sigma = s(0) = \int_{y_1}^{0} \{M^2(y)-1\}^{\frac{1}{2}}\, dy$ and the arbitrary constant has been chosen to satisfy the first of conditions (17).

Now from (27) and (19) we have, retaining only those terms in Π_{0_y} which contain a factor k and discarding terms of lower order, as $|k| \to \infty$,

$$G(k) \sim F(k)\, e^{-2ik\sigma}\, i \operatorname{sgn} k. \tag{28}$$

And so, if the incident wave represents a simple discontinuity in pressure, of magnitude $\gamma p_1 \Delta$ on the characteristic $x + \beta y = 0$, so that

$$f(x+\beta y) = \tfrac{1}{2}\Delta[1 + \operatorname{sgn}(x+\beta y)], \tag{29}$$

we have, as $|k| \to \infty$,

$$F(k) \sim -\frac{i\Delta}{2\pi k} \tag{30}$$

giving, from (28),

$$G(k) \sim \frac{\Delta}{2\pi |k|}\, e^{-2ik\sigma}. \tag{31}$$

Equation (31) shows us that

$$g(x-\beta y) = -\frac{\Delta}{\pi} \log |x - \beta y - 2\sigma| + (\text{continuous function}), \tag{32}$$

indicating that the reflected wave possesses not a simple discontinuity but a logarithmic singularity on the characteristic $x - \beta y - 2\sigma = 0$. Although a positive logarithmic infinity of pressure is postulated in the reflected wave, this cannot be realized physically, and may be interpreted (see Lighthill 1950) as a pressure ridge, that is, as a rapid compression followed by a rapid expansion.

The reflexion of an incident wave of the form given by (29) can perhaps be seen more clearly by examining the pressure coefficient $\varpi(x, y)$ in the supersonic part of the non-uniform region $y_1 < y \leqslant 0$. From equations (18), (27) and (30), we have, as $|k| \to \infty$,

$$\Pi(k, y) \sim \frac{M(y)\{M^2(y)-1\}^{-\frac{1}{4}}}{M_1(M_1^2-1)^{-\frac{1}{4}}} \left[\left(\frac{-i\Delta}{2\pi k}\right) e^{ik(s-\sigma)} + \left(\frac{\Delta}{2\pi |k|}\right) e^{-ik(s+\sigma)}\right], \tag{33}$$

showing that $\varpi(x, y)$ has a discontinuity of magnitude

$$\frac{M(y)\{M^2(y)-1\}^{-\frac{1}{4}}}{M_1(M_1^2-1)^{-\frac{1}{4}}}\Delta \quad \text{on} \quad x = \sigma - s,$$

whilst

$$\varpi(x, y) + \frac{\Delta}{\pi} \frac{M(y)\{M^2(y)-1\}^{-\frac{1}{4}}}{M_1(M_1^2-1)^{-\frac{1}{4}}} \log |x - (\sigma + s)|$$

is continuous on $x = \sigma + s$. The incident wave then, on entering the non-uniform region at $(0, 0)$ is propagated along the characteristic with slope

$$dx/dy = -\{M^2(y)-1\}^{\frac{1}{2}},$$

that is along the line $x = \sigma - s$, until the sonic line $y = y_1$ is reached at $x = \sigma$; the wave is then reflected along the other characteristic through this point, $x = \sigma + s$, emerging from the non-uniform region at the point $(2\sigma, 0)$ and proceeding along the characteristic $x = \beta y + 2\sigma$ as we saw in (32). The singularity of the pressure coefficient ϖ on the sonic line itself may be estimated from equations (18), (25) and (64), since we know that $\Pi_0(k, y_1) \sim f_1(k)\, u_1(y_1) + f_2(k)\, u_2(y_1)$. In this way we see that $\varpi(x, y_1)$ possesses a singularity of the form

$$|x-\sigma|^{-\frac{1}{6}}\left(\operatorname{sgn}(x-\sigma)\cos\frac{5\pi}{12}+\sin\frac{5\pi}{12}\right). \tag{34}$$

It is worth noting here that if the incident wave $f(x)$ consists of one or more simple discontinuities in the pressure gradient, then no infinity of pressure in the reflected wave arises. This can be seen by observing that $\partial\varpi/\partial x$ satisfies the same differential equation and boundary condition, when $y \to -\infty$, as ϖ itself. Thus, if in the incident wave $\partial\varpi/\partial x$ has a discontinuity Δ_1, then the reflected wave will possess a region of logarithmically infinite pressure gradient. Again, if the incident wave possesses a discontinuity $-\Delta_1$ in $\partial\varpi/\partial x$ followed by one of Δ_1 representing a Prandtl–Meyer expansion, then the reflected wave has a region of large negative pressure gradient followed by one of large positive pressure gradient. Integration to find ϖ shows that a Prandtl–Meyer expansion is reflected as an expansion followed immediately by compression.

Although the flow direction is directly deducible from the second of equations (4), it is interesting to investigate it by methods similar to those developed above. Consider, first, the flow in the supersonic part of the non-uniform region when the incident wave is given by (29). In this region the asymptotic form of the Fourier transform of the flow deflexion, given by equations (20), (27) and (30), as $|k| \to \infty$, is

$$H(k,y) \sim \frac{(M_1^2-1)^{\frac{1}{2}}\{M^2(y)-1\}^{\frac{1}{2}}}{M_1\, M(y)}\left\{\frac{i\Delta}{2\pi k}\, e^{-ik(\sigma-s)}+\frac{\Delta}{2\pi|k|}\, e^{-ik(\sigma+s)}\right\}. \tag{35}$$

The first term in (35) indicates that on crossing the characteristic $x = \sigma - s$ (along which the incident wave lies) the flow is deflected by an amount

$$-\Delta\,\frac{(M_1^2-1)^{\frac{1}{2}}\{M^2(y)-1\}^{\frac{1}{2}}}{M_1\, M(y)},$$

agreeing with the weak wave deflexion $-\Delta(M_1^2-1)^{\frac{1}{2}}/M_1^2$ at $y = 0$. The second term in (35) shows that, on crossing the characteristic $x = \sigma + s$, the flow is subjected to a logarithmically infinite deflexion, as may be expected when a logarithmically infinite pressure is involved.

In the subsonic region $y < y_1$, equations (20), (24) and (30) give us, as $|k| \to \infty$,

$$H(k,y) \sim \frac{(M_1^2-1)^{\frac{1}{2}}\{1-M^2(y)\}^{\frac{1}{2}}}{\sqrt{2}\, M_1\, M(y)}\left[\frac{\Delta}{2\pi|k|}+\frac{i\Delta}{2\pi k}\right] e^{-|k|\,|s|}\, e^{-ik\sigma}. \tag{36}$$

Neglecting the factor $e^{-|k|\,|s|}$ for the moment, we see that the fluid is violently disturbed in the neighbourhood of $x = \sigma$, the point on the sonic line at which the incident wave is reflected, as we may have imagined. The effect of the factor $e^{-|k|\,|s|}$ is merely to smooth out, over a distance of the order of $|s|$, the other features present.

Both equations (35) and (36) imply that the flow along the sonic line $y = y_1$ does not undergo any discontinuous change in direction. That this is so can be shown by examining the singularities of $\theta(x, y)$ along the sonic line. From equations (20) and (30) we have

$$H(k, y_1) = \frac{\Delta}{\pi k^2} \Pi_{0_y}(k, y_1) \cos(k\sigma - \tfrac{1}{4}\pi \operatorname{sgn} k) \exp[-ik\sigma + \tfrac{1}{4}\pi i \operatorname{sgn} k]. \qquad (37)$$

Substituting for $\Pi_{0_y}(k, y_1)$ from (25) and (64) in (37), and performing the transformation, shows that on the sonic line $\theta(x, y)$ contains no singularity worse than

$$|x-\sigma|^{\frac{1}{3}} \left(\operatorname{sgn}(x-\sigma) \cos\frac{7\pi}{12} - \sin\frac{7\pi}{12}\right),$$

indicating that the flow along the sonic line suffers no discontinuous change in direction.

5. Solution with model profile

As we have seen, in §4, it is possible to examine the main features of the flow by finding asymptotic forms, as $|k| \to \infty$, for the Fourier transforms of the reflected wave g, the pressure coefficient ϖ and the flow direction θ, and hence picking out the singularities in these quantities. This was achieved without specifying the Mach-number distribution $M(y)$ uniquely. We shall now, using a specific Mach-number distribution, examine the afore-mentioned quantities in more detail.

Let the Mach-number distribution in the non-uniform region be given by

$$M(y) = M_1 e^{Ay}, \qquad (38)$$

where A is a positive constant. This satisfies the conditions imposed on $M(y)$ in §2. Before substituting for $M(y)$ in equation (14) from (38), it is convenient in this case to transform the independent variable in (14) from y to $M(y)$. We then have

$$M'^2(y)\frac{d^2\Pi}{dM^2} + M''(y)\frac{d\Pi}{dM} - 2\frac{M'^2(y)}{M(y)}\frac{d\Pi}{dM} + k^2\{M^2(y)-1\}\Pi = 0, \qquad (39)$$

the primes denoting differentiation with respect to y. Putting $M(y) = M_1 e^{Ay}$, equation (39) becomes

$$\Pi'' - \frac{1}{M}\Pi' + \frac{k^2}{A^2}\left(1 - \frac{1}{M^2}\right)\Pi = 0, \qquad (40)$$

where the primes now denote differentiation with respect to M. If we now put

$$\Pi = z\,u(z), \qquad (41)$$

where
$$z = Mk/A, \qquad (42)$$

the equation satisfied by $u(z)$ is

$$z^2 u'' + zu' + \{z^2 - [1 + (k/A)^2]\}u = 0, \qquad (43)$$

the primes denoting differentiation with respect to z. Equation (50) is Bessel's equation, and hence the solution of (40) satisfying conditions (17) is

$$\Pi_0(k, y) = C\,e^{Ay} J_{\sqrt{[1+(k/A)^2]}}(M_1 e^{Ay} k/A), \qquad (44)$$

where
$$C = [J_{\sqrt{[1+(k/A)^2]}}(M_1 k/A)]^{-1}. \qquad (45)$$

In the general case of an arbitrary Mach-number profile it has not been found possible to say anything about the upstream influence of the disturbance, i.e. how far upstream of the characteristic $x+\beta y = 0$, along which the incident wave lies, the disturbance in the non-uniform region penetrates. In the analagous problem of shock-wave-boundary-layer interaction, Lighthill (1953) has shown that, even for non-separating boundary layers, the upstream influence of a shock wave extends some tens of boundary-layer thicknesses ahead of the point of incidence. It will be shown that in the present case, when the Mach-number variation across the non-uniform region is given by equation (38), the upstream influence is negligible, as we may expect in the absence of a wall.

Since $f(x) = 0$ for $x < 0$, the transform $F(k) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$ is regular in the lower half-plane. Then the only singularities of $\Pi(k,y)$ are poles at the zeros of the denominator of equation (18). Hence, whenever $x < 0$, the function $\varpi(x,y)$ may by (11) be expressed as $-2\pi i$ times the sum of the residues of $\Pi(k,y)\,e^{ikx}$ at the zeros of the denominator of (18) in the lower half-plane. Since only the lower half-plane is to be considered it is convenient to put $k = -iK$, also, as it is unlikely that the upstream disturbance will be wavy, we shall confine ourselves to real and positive K.

Before proceeding further we shall now introduce a length δ, to be known as the 'mixing region thickness', such that at $y = -\delta$ the velocity has fallen to only 1 % of its free-stream value M_1. Thus,

$$M_1\,e^{-A\delta} = 0{\cdot}01 M_1,$$

giving

$$A \simeq 4{\cdot}61/\delta. \tag{46}$$

Now, since

$$\varpi(x,y) = -2\pi i \sum_{n=1}^{\infty} R_n(y)\,e^{K_n x},$$

the smallest K_n ($= K_1$, say), will give an estimate of how far upstream the disturbance penetrates. Defining $\alpha = K_1/A$, equation (46) shows that

$$e^{K_1 x} = \exp\left[(4{\cdot}61\alpha/\delta)\,x\right], \tag{47}$$

and hence, if $\alpha \geqslant 1$, equation (47) shows that the disturbance upstream of the incident wave is effectively damped out within one 'mixing region thickness'. This means that to show the upstream influence is negligible is to show that no zeros of the denominator of equation (18) lie in the range $0 \leqslant K/A < 1$. We are concerned then with solutions of the equation

$$\Pi_{0_y}(-iK,0) + K\beta\Pi_0(-iK,0) = 0, \tag{48}$$

where $\Pi_0(k,y)$ is given by equation (44). On putting $k = -iK$, equation (44) becomes

$$\Pi_0 = C_1\,e^{Ay} I_{\sqrt{[1-(K/A)^2]}}(M_1\,e^{Ay}K/A), \tag{49}$$

where

$$C_1 = [I_{\sqrt{[1-(K/A)^2]}}(M_1 K/A)]^{-1}. \tag{50}$$

If we now put

$$[1-(K/A)^2]^{\frac{1}{2}} = \nu, \quad M_1 K/A = \eta, \tag{51}$$

substitution of (49) into (48) gives, as the equation to be solved,

$$1+\eta\frac{I_\nu'(\eta)}{I_\nu(\eta)}+\frac{\eta\beta}{M_1}=0, \tag{52}$$

or

$$1+\nu+\eta\frac{I_{\nu+1}(\eta)}{I_\nu(\eta)}+\frac{\eta\beta}{M_1}=0. \tag{53}$$

Now, as has been explained above, the object is not to obtain explicit solutions of equation (53) but is rather to obtain the negative result that no solution exists in the range $0 \leqslant K/A < 1$, i.e. in $0 \leqslant \eta < M_1$. It can be shown that

$$\frac{I_{\nu+1}(\eta)}{I_\nu(\eta)}=\eta^2\sum_{m=0}^{\infty}a_m\eta^{2m}\Big/\sum_{m=0}^{\infty}b_m\eta^{2m}, \tag{54}$$

where both $a_m, b_m > 0$; hence, in the range of η under consideration, the last three terms of (53) are never less than zero, and so equation (53) has no roots in this range. We may conclude that the upstream influence is negligibly small.

Although analytic solutions of equation (14) have been found when the Mach-number distribution is given by (38), it has not been found possible to integrate equation (12) analytically to obtain the reflected wave. However, using numerical methods in conjunction with a high-speed electronic computer, the form of the reflected wave, for a particular free-stream Mach number $M_1=\sqrt{2}$, has been found both when the incident wave is a simple discontinuity representing a shock wave and when the pressure gradient of the incident wave contains simple discontinuities representing a Prandtl–Meyer expansion fan. For an incident shock wave, the reflected wave was expressed, from (12) and (19), as

$$\Delta^{-1}g(x)+\tfrac{1}{2}=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{i(k/A)Ax}\frac{i\Pi_{0Y}(k/A,0)+(k/A)\beta\Pi_0(k/A,0)}{\Pi_{0Y}(k/A,0)+i(k/A)\beta\Pi_0(k/A,0)}\frac{1}{k/A}\,d(k/A), \tag{55}$$

where $Y = Ay$ and $F(k)$ is, from (29), given by

$$F(k)=\frac{\Delta}{2}\left\{\delta(k)-\frac{i}{2\pi k}\right\}, \tag{56}$$

where $\delta(x)$ is the Dirac delta function such that $\delta(x) = 0$, $x \neq 0$ and

$$\int_{-x_0}^{x_0}f(x)\,\delta(x)\,dx=f(0)\quad\text{for any } x_0>0.$$

The function $\Pi_{0Y}(k/A,0)$, occurring in (55), was obtained by numerically integrating (14) with the Mach-number distribution given by (38), taking $M_1=\sqrt{2}$. The reflected wave thus obtained is given in figure 1, where $\Delta^{-1}g(x)$ is plotted against Ax, showing the very narrow region of compression followed by a broader region of expansion, the overall drop in pressure being of magnitude $\gamma p_1\Delta$. In figure 2 the reflected wave, due to an incident shock wave, both for the Mach-number distribution given by (38) and for the distribution $M(y)=M_1e^{-A^2y^2}$ with $M_1=\sqrt{2}$, are compared by plotting $\Delta^{-1}g(x)$ against x/δ, where δ is the mixing region thickness. The comparison shows that the form of $M(y)$, beyond certain basic requirements, is not critical in determining the reflected wave. If the incident wave is a Prandtl–Meyer expansion fan, represented by a discontinuity $-\gamma p_1 \Delta/l$ in the pressure gradient followed by a further discontinuity $\gamma p_1 \Delta/l$, where l is the length of the expansion and the overall drop in pressure is $\gamma p_1 \Delta$, then, as we have seen earlier, since $\partial \varpi/\partial x$ satisfies the same equation and boundary conditions as ϖ, the pressure gradient in the reflected wave is given as a linear combination of two reflected waves of the type shown in figure 1. The

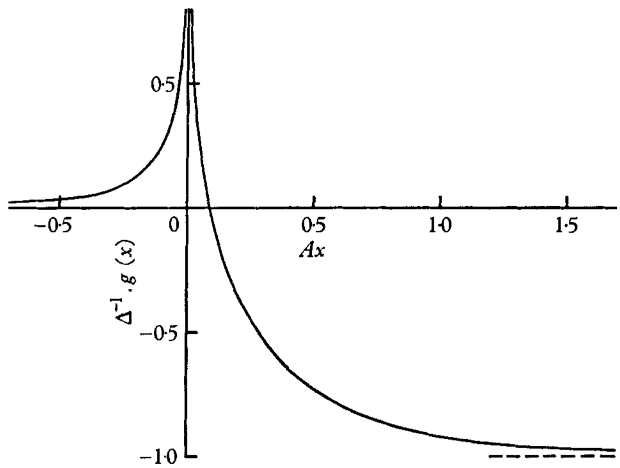

FIGURE 1. Reflected wave when the incident wave represents a shock wave.

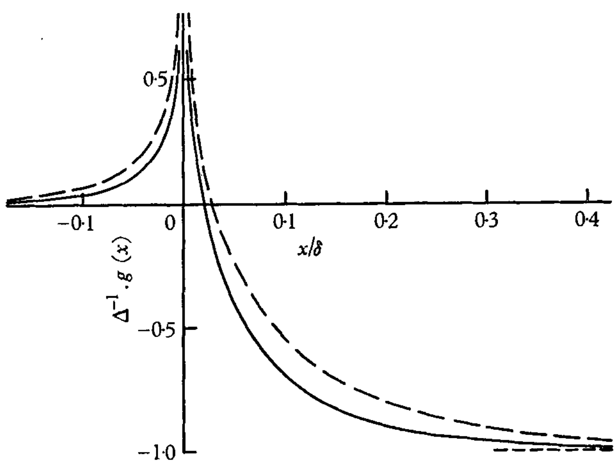

FIGURE 2. Comparison of the reflected wave due to an incident shock wave when (a) $M(y) = \sqrt{2}e^{Ay}$, ———; (b) $M(y) = \sqrt{2}e^{-A^2y^2}$, – – – –.

reflected wave is then obtained by simple integration. The reflected wave due to such an incident wave in the special case $Al = 1$ is shown in figure 3. This shows that there is, initially, a slight further drop in pressure followed by a sharp rise in pressure, the overall pressure rise being $\gamma p_1 \Delta$.

The Schlieren photograph shown in figure 4 (plate 1) is an example in which a plane shock wave, strong enough to cause separation, is incident on a boundary layer. The boundary layer is fully separated at the point where the incident shock meets it, and the wave reflected there is seen to consist of a shock wave preceding a broad expansion wave. This shock wave loses its identity when it becomes coincident with the main reflected wave. Further Schlieren photographs (Johannesen 1957) show an axisymmetric jet emerging from a nozzle at a pressure less than atmospheric; in these the shock wave starting at the nozzle is incident on the mixing region and is reflected mainly as an expansion wave. On closer examination, however, remembering that we are looking through an axisymmetric jet, a thin dark line representing a narrow region of compression is seen immediately before the main expansion wave. Although the theory given above is for the two-dimensional case, we may expect the results, including the presence of such a region of compression, to hold qualitatively in the axisymmetric problem. These photographs do then provide evidence in favour of the theory given above.

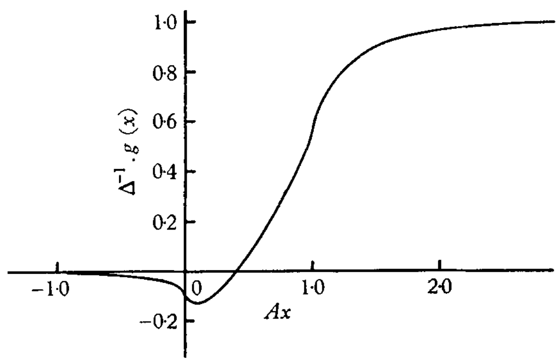

FIGURE 3. Reflected wave when the incident wave represents an expansion fan.

6. Improved linear theory

All the results obtained so far in preceding sections have been obtained using ordinary linearized theory; that is, in a uniform two-dimensional supersonic flow in which the two sets of characteristics are two sets of parallel straight lines, discontinuities to represent incident waves have been fitted on one or more of these characteristics pointing upstream, giving rise to a reflected wave situated along other characteristics pointing downstream, all the characteristics remaining unchanged. This failure of linear theory is fundamental, since regions in which shock waves occur are regions where the characteristics have run together and overlapped to form a limit line. The method by which we shall find the position of the shock wave in such a region is given in appendix B, which should be read in conjunction with the following discussion. For simplicity we shall only consider the flow around the reflected waves, shown in figures 1 and 3, and only in the region of uniform flow. The supersonic part of the non-uniform region will be very thin and errors introduced by ignoring it will only be significant at large distances from the non-uniform region. Furthermore, since all physical quantities have been obtained by neglecting products of small quantities, we shall here neglect such terms by using an approximation to $\tan(\mu+\theta)$, although the method outlined in appendix B allows products of small quantities to be retained.

Thus, taking

$$\varpi = \Delta\,\mathrm{sgn} f + g(x-\beta y), \tag{57}$$

where $\operatorname{sgn} f = +1$ if the incident wave represents a shock wave with pressure rise $\gamma p_1 \Delta$, and $\operatorname{sgn} f = -1$ if the incident wave represents a Prandtl–Meyer expansion with overall drop in pressure $\gamma p_1 \Delta$, we have, from (4),

$$\theta = \frac{1}{M_1^2}\int_x^\infty \frac{\partial \varpi}{\partial y}\,dx,$$

hence

$$\theta = \frac{\beta}{M_1^2}\{\Delta \operatorname{sgn} f + g(x-\beta y)\}. \tag{58}$$

Also, from Bernoulli's equation, neglecting products of small quantities, it can be shown that

$$\mu = \mu_\infty + \frac{(\gamma-1)\,M_1^2+2}{2M_1^2(M_1^2-1)^{\frac{1}{2}}}\varpi. \tag{59}$$

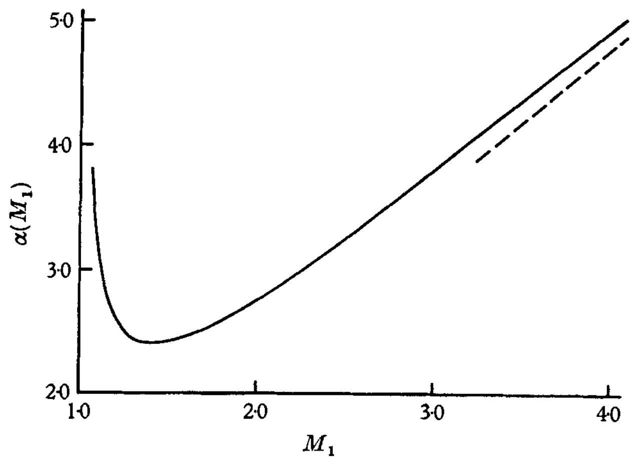

FIGURE 5. The coefficient $\alpha(M_1)$ occurring in equation (60) ($\gamma = 1{\cdot}4$).

Thus, along the correct characteristic $z(x,y) = \text{const.}$, we have from (58), (59) and (65)

$$\frac{dx}{dy} = \beta - \alpha\{\Delta \operatorname{sgn} f + g(z)\} \tag{60}$$

approximately, where $\alpha = \frac{1}{2}\beta^{-1}\{(\gamma-1)\,M_1^2+2\}+\beta$ is shown in figure 5. If the value of z on a characteristic is taken as the value of $(x-\beta y)$ at the point where the characteristic meets the line $y = 0$, equation (60) can be integrated to give

$$x = \beta y - \alpha\{\Delta \operatorname{sgn} f + g(z)\}\,y + z. \tag{61}$$

Having found the characteristics, the mass-flow function $h(\mu)$ can be most easily determined along a line $x = \text{const.}$ by observing the value of μ at each point of the line where it is cut by a characteristic. Figures 6 and 7 show the mass flow function at different stations when the reflected wave is due to an incident shock and Prandtl–Meyer fan, respectively, with $\Delta = 0{\cdot}1$. In each case the shock wave is supposed to emerge from the non-uniform region at the point (0, 0). The associated flow patterns in figures 8 and 9 show very clearly the regions of compression, where the characteristics close together, and the regions of expansion where the characteristics spread out fanwise. It is important to note that now, in the reflected wave due to an incident shock wave, the only point where the pressure is infinite is at (0, 0), elsewhere along the reflected shock the rise in pressure is finite, decreasing to zero under the influence of the neighbouring expansion fan. This removes the main peculiarity of the original theory, and makes what is left more satisfactory and credible.

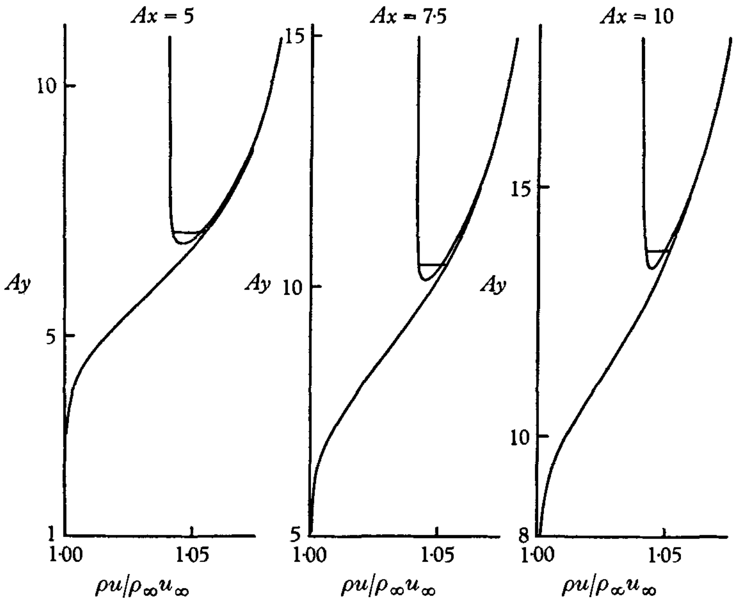

FIGURE 6. The mass flow at different stations when the incident wave is a shock wave. The horizontal line cutting off lobes of equal area indicates the position of the reflected shock wave ($M_1 = \sqrt{2}$, $\Delta = 0 \cdot 1$).

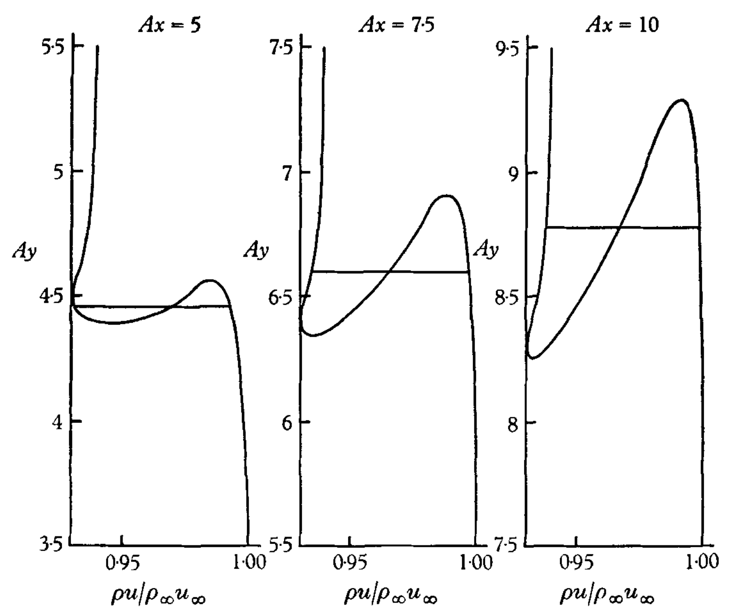

FIGURE 7. The mass flow at different stations when the incident wave is an expansion fan. The horizontal line cutting off lobes of equal area indicates the position of the reflected shock wave ($M_1 = \sqrt{2}$, $\Delta = 0 \cdot 1$).

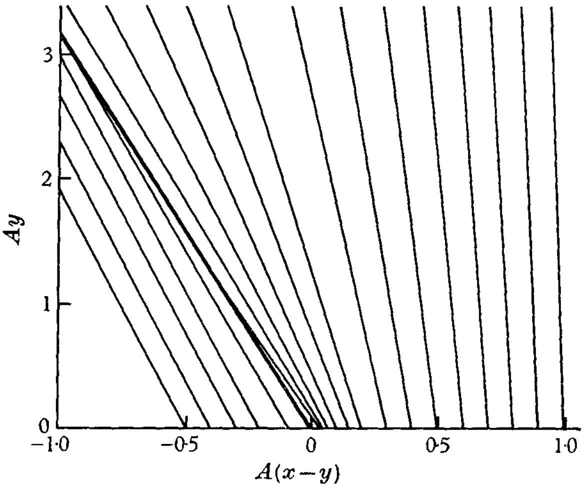

FIGURE 8. The flow pattern for the reflected wave when the incident wave is a shock wave. The shock wave in the reflected wave is indicated by the thicker line ($M_1 = \sqrt{2}$, $\Delta = 0{\cdot}1$).

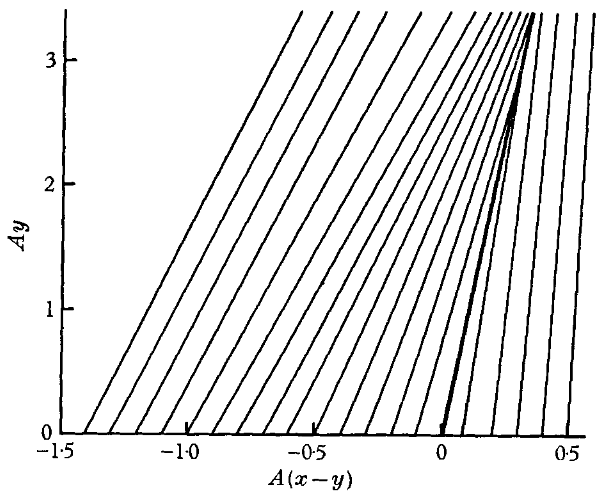

FIGURE 9. The flow pattern for the reflected wave when the incident wave is an expansion fan. The shock wave in the reflected wave is indicated by the thicker line ($M_1 = \sqrt{2}$, $\Delta = 0{\cdot}1$).

Appendix A

The results of Langer's theory used are briefly summarized here.

In the equation
$$\frac{d^2u}{dy^2}+p(y)\frac{du}{dy}+\{k^2q(y)+r(y)\}u = 0,$$
let $p(y)$, $q(y)$ and $r(y)$ be twice differentiable in $y_0 \leqslant y \leqslant y_2$. Let $q(y) < 0$ in $y_0 \leqslant y < y_1$ and $q(y) > 0$ in $y_1 < y \leqslant y_2$, while $q'(y_1) > 0$. Then two asymptotic solutions, valid in $y_0 \leqslant y \leqslant y_2$ as $|k| \to \infty$ for $y \neq y_1$ are

$$\left.\begin{aligned} u_1(y) &\sim \exp\left(-\frac{1}{2}\int_{y_1}^{y} p\,dy\right)\frac{|s|^{\frac{1}{6}}}{|q|^{\frac{1}{4}}}\left[\frac{e^{iks}}{(iks)^{\frac{1}{3}}}+\frac{e^{-iks}}{(-iks)^{\frac{1}{3}}}\right],\\ u_2(y) &\sim \exp\left(-\frac{1}{2}\int_{y_1}^{y} p\,dy\right)\frac{|s|^{\frac{5}{6}}}{|q|^{\frac{1}{4}}}\operatorname{sgn}(y-y_1)\left[\frac{e^{iks}}{(iks)^{\frac{5}{3}}}+\frac{e^{-iks}}{(-iks)^{\frac{5}{3}}}\right],\end{aligned}\right\} \qquad (62)$$

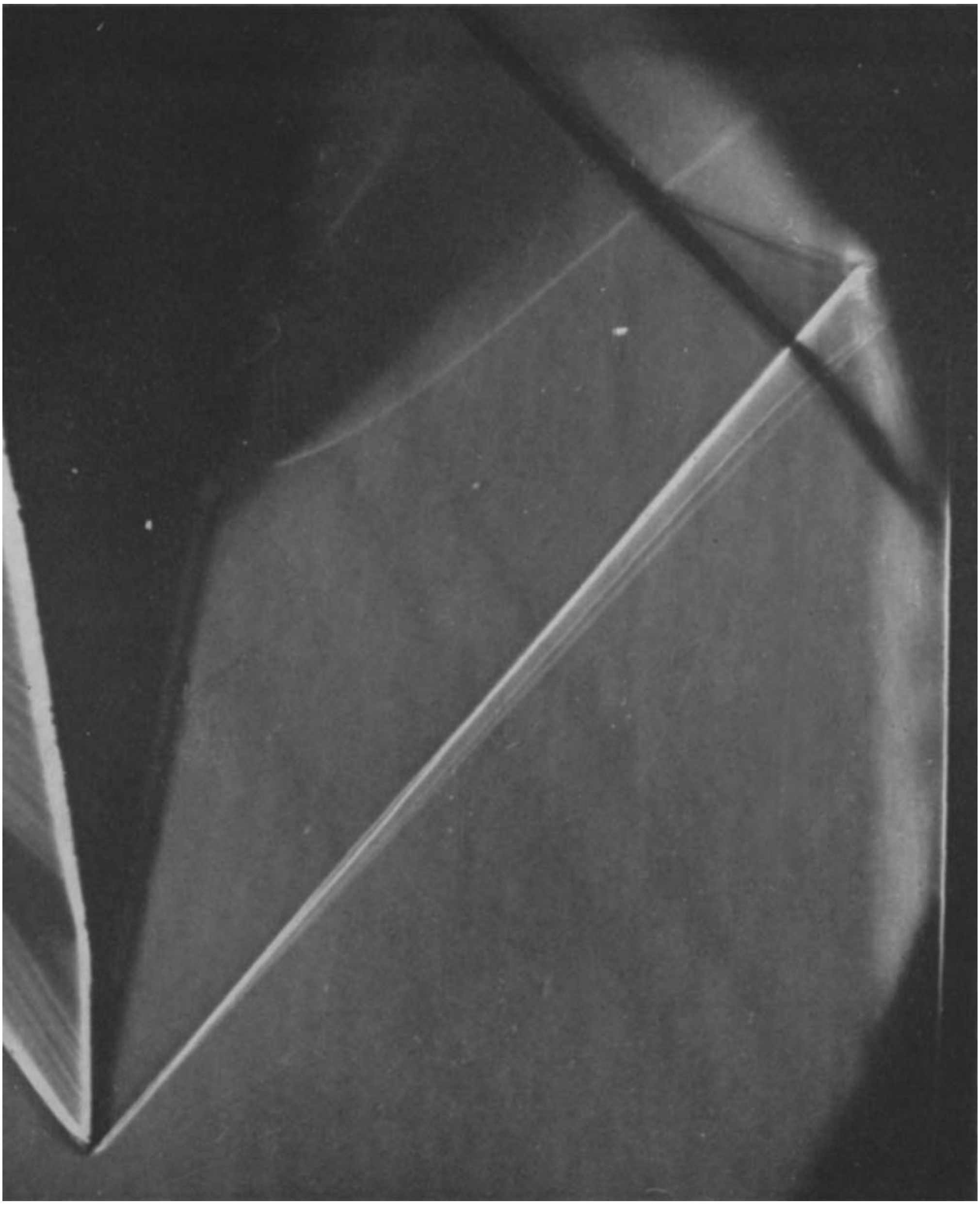

FIGURE 4 (plate 1). Shock wave incident on a two-dimensional boundary layer causing separation.

where

$$s = \int_{y_1}^{y} q^{\frac{1}{2}}\, dy. \tag{63}$$

When $y = y_1$

$$\left.\begin{aligned} u_1(y_1) &= \left(\frac{2}{3q'(y_1)}\right)^{\frac{1}{6}} \frac{2^{\frac{1}{3}}(2\pi)^{\frac{1}{2}}}{(-\frac{1}{3})!}, \quad u_1'(y_1) = 0, \\ u_2(y_1) &= 0, \quad u_2'(y_1) = (\tfrac{2}{3})^{\frac{5}{6}} \{q'(y_1)\}^{\frac{1}{6}} \frac{2^{-\frac{1}{3}}(2\pi)^{\frac{1}{2}}}{(\frac{1}{3})!}. \end{aligned}\right\} \tag{64}$$

If a special asymptotic solution is determined in either of these subintervals then it can be expressed in terms of u_1 and u_2 and hence extended into the other subinterval. In the theory of §4, $y = y_1$ is seen to correspond to the sonic line and the method described here is used to extend solutions of (14) across this line.

Appendix B

The main fault that Whitham (1952) finds with linear theory is that although it gives a correct first approximation everywhere for any physical quantity on a characteristic, the characteristics themselves are incorrectly placed. From the physical quantities given by linear theory, Whitham calculates a first approximation to the correct characteristics, and, using the condition that when products of small quantities are negligible the shock bisects the pair of characteristics which meet it at any point, shows how the flow pattern can be completely determined. More refined methods of determining the flow pattern have been described by Lighthill (1957) and Kantrowitz (1958), and the method outlined below is foreshadowed by both these authors. The method, like Whitham's, can be criticized for assuming the discontinuous character of the solution in advance, but, since the only approximation to be made is that the flow is isentropic, the method is more accurate than Whitham's.

Assuming isentropic flow, we have that on the characteristics which point downstream in a two-dimensional flow

$$\frac{dy}{dx} = \tan(\mu + \theta), \tag{65}$$

where μ is the local Mach angle. Since, in isentropic flow, μ is constant along a characteristic and θ is a function of the Mach angle we can write (65) as

$$y - x\tan(\mu + \theta) = \text{const.} \tag{66}$$

Consider now the flux of mass parallel to the x-axis

$$\rho u/\rho_\infty u_\infty = \rho q \cos\theta/\rho_\infty q_\infty, \tag{67}$$

the subscript ∞ referring to some standard state of the flow. Now $\theta = f(\mu) - f(\mu_\infty)$ and $\rho q/\rho_\infty q_\infty = F(\mu, \mu_\infty)$ and so we may write (67) as

$$\rho u/\rho_\infty u_\infty = h(\mu, \mu_\infty), \tag{68}$$

and $h(\mu, \mu_\infty)$ will be known as the 'mass-flow function'. It may be shown that for a given μ_∞ the mass-flow function is an increasing function of μ, and when $\mu_\infty = \frac{1}{4}\pi$ figure 10 shows h as a function of μ.

Now, since on a characteristic

$$y - x\tan(\mu+\theta) = y_0 - x_0\tan(\mu+\theta), \tag{69}$$

it follows that

$$h(x,y) = h(x_0, y_0). \tag{70}$$

From (69) we may write equation (70) as

$$h(x,y) = h[x_0, y + (x_0 - x)\tan(\mu+\theta)], \tag{71}$$

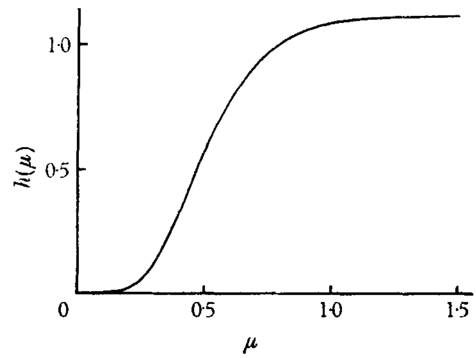

FIGURE 10. Mass flow function $h(\mu)$ when $\mu_\infty = \frac{1}{4}\pi$.

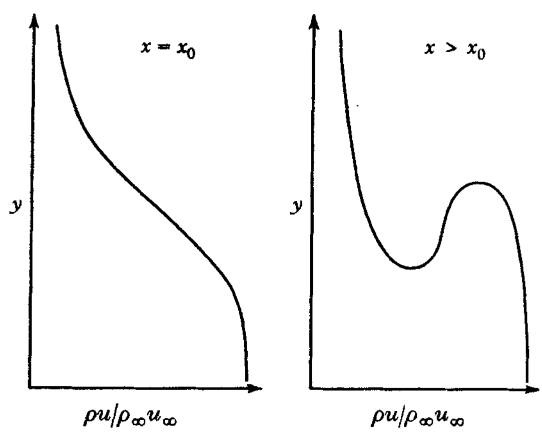

FIGURE 11. Typical mass-flow curve showing breakdown of uniqueness.

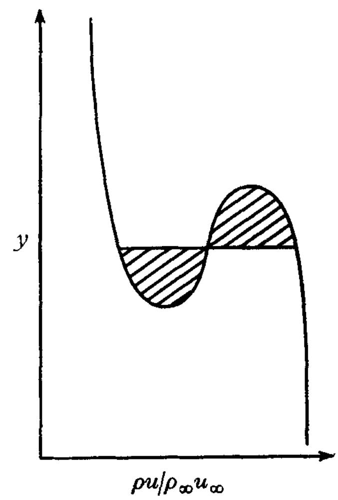

FIGURE 12. Typical non-unique mass-flow curve with horizontal line cutting off lobes of equal area indicating the position of the shock wave.

and so if the mass-flow function is known at any station x_0, for all y, then equation (71) enables us to find it at any other station x. Since μ is an increasing function of the pressure, equation (71) indicates that if a disturbance to the uniform flow is compressive, and if at x_0 the mass-flow function is a single-valued function of y, for some $x > x_0$ uniqueness breaks down and the mass-flow function becomes multi-valued as in figure 11. Obviously this state of affairs cannot exist physically, and a discontinuity representing a shock wave must be inserted on the mass-flow curve in order that the mass flow may remain single valued. Since the area beneath the curve represents the mass flow, the discontinuity must be introduced so as to leave the total area under the curve constant, because the total mass flow cannot change as a result of the appearance of a shock wave. As shown in figure 12 this is achieved by letting the discontinuity cut off lobes of equal area on the mass-flow curve. The shock wave first develops where the tangent at the point of inflexion of the mass-flow curve becomes perpendicular to the y-axis under this shearing process.

Whitham's theory may be easily deduced from the above by noting that if we neglect products of small quantities then we can write

$$\tan(\mu+\theta) = \tan\mu_\infty + K(h-1), \tag{72}$$

and so, if the mass-flow function is known at any x, then, as equation (72) shows, it may be deduced at any other value of x by a simple linear shearing process. Since this shearing process conserves areas it may be noted that if, on the mass-flow function curve even when single-valued, lines are inserted cutting off equal area lobes on the curve then further downstream, each of these lines, in turn, will become perpendicular to the y-axis cutting off equal area lobes as we require. And so if, on a given mass-flow curve, lines are drawn cutting off lobes of equal area, then the complete flow pattern can be deduced since the pair of characteristics which pass through the points corresponding to the end-points of each line meet on the shock wave. This is, in effect, the rule given by Whitham (1952).

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